an eigensubspace by the substitution $\boldsymbol{\theta}_R = \mathbf{Q}\boldsymbol{\theta}_k$ where $\mathbf{Q}$ is the matrix which diagonalizes $\mathbf{A}$.

$$\begin{aligned}\boldsymbol{\theta}_R^\dagger \mathbf{A}\boldsymbol{\theta}_R &= \boldsymbol{\theta}_k^\dagger \mathbf{Q}^\dagger \mathbf{A}\mathbf{Q}\boldsymbol{\theta}_k \\ &= \boldsymbol{\theta}_k^\dagger \mathbf{D}\boldsymbol{\theta}_k \\ &= \lambda_k |\boldsymbol{\theta}_k|^2\end{aligned} \tag{6}$$

We can find the eigenvalues $\lambda_k = \omega_k^2$ by considering the approach outlined in Mazenko [1]. We begin by defining

$$\begin{aligned}\phi(\mathbf{R}_i, \mathbf{R}_j) &\equiv \frac{\partial}{\partial\theta(\mathbf{R}_i)}\frac{\partial}{\partial\theta(\mathbf{R}_j)} H \\ &= \frac{S^2}{4}\sum_{\mathbf{R}_k,\mathbf{R}_l} J_{\mathbf{R}_k\mathbf{R}_l}\frac{\partial}{\partial\theta(\mathbf{R}_i)}\frac{\partial}{\partial\theta(\mathbf{R}_j)}[\theta(\mathbf{R}_k)-\theta(\mathbf{R}_l)]^2 \\ &= JS^2\sum_{\mathbf{a}}\left[\delta_{\mathbf{R}_i,\mathbf{R}_j} - \delta_{\mathbf{R}_i+\mathbf{a},\mathbf{R}_j}\right]\end{aligned} \tag{7}$$

Using this result, we can now find the dynamical matrix

$$D(\mathbf{R}_i, \mathbf{R}_j) \equiv \delta_{\mathbf{R}_i,\mathbf{R}_j}\sum_{\mathbf{R}_l}\phi(\mathbf{R}_i,\mathbf{R}_l) - \phi(\mathbf{R}_i,\mathbf{R}_j)$$

$$D(\mathbf{R}) = JS^2\left[\sum_{\mathbf{a}}\delta_{\mathbf{R},\mathbf{a}} - 2d\delta_{\mathbf{R},\mathbf{0}}\right] \tag{8}$$

where d is the dimension of the system, and we have assumed a hypercubic lattice. Taking the Fourier transform of Equation 8 we obtain our dispersion relation

$$\begin{aligned}\omega_{\mathbf{k}}^2 &= D(\mathbf{k}) = 2JS^2 d\,[1-\cos(\mathbf{k}\cdot\mathbf{a})] \\ &= 4JS^2 d\sin^2\left(\frac{\mathbf{k}\cdot\mathbf{a}}{2}\right) \\ &\simeq JS^2 d(\mathbf{k}\cdot\mathbf{a})^2, \quad \text{for k small}\end{aligned} \tag{9}$$

Thus we see that ω_k depends linearly on k and the constant of proportionality, which represents the phase velocity of the spin wave is given by

$$c_s = \sqrt{Jd}Sa \sim \sqrt{J} \tag{10}$$

That is, the wave speed goes as the square root of the exchange coupling energy. We can now rewrite our Hamiltonian

$$\begin{aligned}H &= H_o + H_\theta \\ &= -\frac{S^2\gamma NJ}{4} + \frac{1}{2}\sum_k \omega_k^2|\boldsymbol{\theta}_k|^2\end{aligned} \tag{11}$$

setting $A = e^{-\beta H_0}$ the associated partition function is then

$$\begin{aligned}Q_N &= A\int_0^\infty d^N\theta_k\, e^{-\frac{\beta}{2}\sum_{k=1}^\infty \omega_k^2|\boldsymbol{\theta}_k|^2} \\ &= A\prod_k\sqrt{\frac{2\pi k_b T}{\omega_k^2}}\end{aligned} \tag{12}$$

LONG RANGE SPIN CORRELATIONS

We can evaluate long range spin correlations in our system by computing the mean value of spin-spin interactions between different locations in our lattice.

$$\begin{aligned}\mathcal{C}(\mathbf{R}) &= \langle \mathbf{S}_{\mathbf{R_1}}\cdot\mathbf{S}_{\mathbf{R_2}}\rangle \\ &= S^2\left\langle\cos[\theta(\mathbf{R})-\theta(\mathbf{R}')]\right\rangle \\ &= S^2\Re\mathrm{e}\left\langle e^{i[\theta(\mathbf{R})-\theta(\mathbf{R}')]}\right\rangle\end{aligned} \tag{13}$$

The evaluation of this average may be determined by taking a cumulant expansion [2] of the form

$$\begin{aligned}\left\langle e^{\lambda U}\right\rangle &= \exp\left[i\,\mathbb{C}_1(U) - \frac{1}{2}\mathbb{C}_2(U) + \ldots\right] \\ &= \exp\left[i\langle U\rangle - \frac{1}{2}\left\langle[U-\langle U\rangle]^2\right\rangle + \cdots\right]\end{aligned} \tag{14}$$

The cumulant is an expansion in statistical moments about the distribution mean. The third and fourth cumulants, measuring the skewness and kurtosis of the distribution respectively, as well as higher order moments, are zero for symmetric potentials and a Gaussian distribution about the mean. If one applies a magnetic field, however, higher moments may have to be considered. Taking $U = \Delta\theta = \theta(\mathbf{R}) - \theta(\mathbf{R}')$ and noting that $\langle U\rangle$ is identically zero assuming a symmetric confining potential of the spins, we get to the second order of the cumulant expansion,

$$\mathcal{C}(\mathbf{R}) = S^2\Re\mathrm{e}\,\exp\left[-\frac{1}{2}\left\langle(\Delta\theta)^2\right\rangle\right]. \tag{15}$$

We can evaluate $< (\Delta\theta)^2 >$ by expanding $\theta(R)$ in Fourier space [3] and referring to our formulation of the partition function (Eq. 12). Taking $\theta(\mathbf{R}) = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}}\theta_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{R}}$ we get

$$\begin{aligned}\left\langle(\Delta\theta)^2\right\rangle &= \left\langle(\theta(\mathbf{R})-\theta(\mathbf{R}'))^2\right\rangle \\ &= \left\langle\frac{1}{N}\sum_{kk'}\left(e^{i\mathbf{k}\cdot\mathbf{R}} - e^{i\mathbf{k}\cdot\mathbf{R}'}\right)\theta_{\mathbf{k}}\left(e^{-i\mathbf{k}'\cdot\mathbf{R}} - e^{-i\mathbf{k}'\cdot\mathbf{R}'}\right)\theta_{\mathbf{k}'}^*\right\rangle \\ &= \frac{2}{N}\sum_k\left(1-\cos\left[\mathbf{k}\cdot(\mathbf{R}-\mathbf{R}')\right]\right)\left\langle|\theta_k|^2\right\rangle \\ &\Rightarrow 2\,\Re\mathrm{e}\left\{\int d\mathbf{k}\left(1-e^{i\mathbf{k}\cdot\mathbf{R}}\right)\left\langle|\theta_k|^2\right\rangle\right\}\end{aligned} \tag{16}$$

where in the last step we have referenced the variance with respect to the spin vector located at the origin ($\mathbf{R'} = 0$). We know from our partition function (Eq. 12) that

$$E_\theta(k) = \frac{1}{2}\omega_k^2|\theta_k|^2,$$

so by the equipartition theorem

$$\langle E_\theta(k)\rangle = \frac{1}{2}\omega_k^2\left\langle|\theta_k|^2\right\rangle = \frac{1}{2}k_B T.$$